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THIS JOURNAL IS DEDICATED TO THE FOLLOWING AIMS: (1) Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values. (2) To supply an additional medium for the publication of expository mathematical articles. (3) To promote more scientific methods of teaching mathematics. (4) To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

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2. The name of the Chairman of each committee is the first in the list of the committee.

3. All manuscripts should be worded exactly as the author wishes them to appear in the MAGAZINE.

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A Challenge

The letter quoted below in full from Mr. Raymond C. Reese, consulting engineer, offers, in effect, a challenge to mathematicians, a challenge, however, which is phrased in terms of courtesy and of generous concession. Many of the questions raised are familiar enough to those of the mathematical fraternity who are sensitive to the service obligations of their science.

TOLEDO, OHIO, March 24, 1943.

Mr. S. T. Sanders,
National Mathematics Magazine,
Baton Rouge, Louisiana.

DEAR SIR:

I have been very much interested in the progress you are making in keeping the NATIONAL MATHEMATICS MAGAZINE functioning. I only subscribed to this a few months ago in response to a letter directed to me at the University of Toledo. In looking through the few copies I have received, an idea came to mind that might or might not be of interest to you. As you know engineering is becoming more and more a matter of applied mathematics. The modern engineer uses in his daily work more advanced mathematics than his forebears had ever heard about. Most engineers are relatively weak in higher mathematics; your magazine probably would not appeal directly to them. It is rather theoretically couched in the terms of the mathematician and relatively difficult for a practicing engineer to follow.

It might be possible to develop a section of applied mathematics where the engineer could get some useful hints particularly applications arranged in a form and in language that he could quickly understand. Just to indicate a few types of problems that have come up:

1. The solution of quadratic equations so common in engineering problems where one of the correctly obtained roots is not a tenable solution, especially the whys and wherefores.

2. Practical methods for solving cubic equations.

3. The solution of simultaneous linear equations involving many unknowns. The method of progressive elimination by detached coefficients, frequently attributed to Gauss, in practical application frequently results in widely varying coefficients. Some will be small decimal fractions, others six or seven digit integers and the question arises how to obtain a reasonable degree of precision without carrying excessive significant figures.

4. The length of a parabolic arc usually gives trouble, especially since all of the other functions of the parabola are relatively simple and easily obtained.

5. In structural engineering the functions of a beam, viz. load curve, shear curve, moment diagram, slope, and deflection curves, represent successive integrations (or differentiations) and the Europeans have developed many useful facts and relationships for various loading conditions that are not so well known in this country.

6. The routine for determining bending moments in continuous beams by graphical construction is quite well developed, but a mathematical demonstration of why the process is sound has never appeared.

7. A similar graphical analysis of rigid frames once appeared in a publication of the University of Bruno, but has never been translated nor has the mathematical background been demonstrated nor has any use been made of the suggestion in this country.

From my own experience I am sure that your publication would have a much greater personal appeal if it dealt in part with practical applications rather than generalizations. I realize that your principal clientele must be pure mathematicians and the present set up would seem to fill their needs admirably. The practicing engineer, however, even though he realizes the need for further mathematical background, does not have too much time for reading and if the material is too highly generalized, and the immediate use is not apparent, he rapidly loses interest. This is by no means a criticism of your excellent work; you have had a number of years experience and this suggestion may not appeal, but after some twenty odd years of teaching young engineers and also of using engineering graduates in design work, I have a very clear feeling that the pure mathematician pursues his own interests with no particular thought to the practical applications and that a good deal of recent mathematical development is not available to the average engineer because of his inability to find a common meeting ground with the mathematician. It is undoubtedly a big order to bridge this gap, but I can think of no more worth while project than to put the methods and analyses of pure mathematics to practical use through the medium of the engineers.

If you see any merit in these suggestions I would be glad to try to summarize a more complete schedule of some of the difficulties the engineer experiences and it may be that your staff would then be interested in shedding some light on these points.

Very truly yours,

RAYMOND C. REESE,
Consulting Engineer.

After deliberation and some correspondence with our editorial colleagues we have decided that the issues so frankly stated by Mr. Reese are worth the attention of our MAGAZINE readers. A few of our editor-colleagues, to whom copies of the letter had been sent, have already expressed themselves.

Colonel William E. Byrne, chairman of the Analysis committee promptly writes:

"I have no objection to 'applied' problems. .... I do not consider it practical to devote a separate department, or its equivalent, to engineering problems, but I do feel that mathematically sound expository papers of high order should be welcomed with the understanding that correct mathematical terminology will be required."

Professor C. D. Smith, representing the Committee on Statistics, includes in his reply the following:

"I do not believe the MAGAZINE should open an engineering department now. As it now stands, the Statistics department, including finance papers, is the *applied mathematics* section of the MAGAZINE. We should bring engineering applications into it. Why not change the name of the Statistics Department to Department of Applied

Mathematics and add a good engineer with theoretical and engineering research interests to our staff to help handle engineering papers? .... Hope you can make something of this letter from Reese."

From Professor N. A. Court, member of the MAGAZINE Problem department and chairman of its "Notes and Comments" is the following definite pronouncement:

"Now as to the suggestion of Raymond Reese. I am fully and enthusiastically for it, for more reasons than one. In the first place, I do not see what objections there can be to it. If Brown University, and other institutions, can devote much time and effort to promote applied mathematics, it is time that the same should be done on the collegiate level. The sooner we get to it the better. On the other hand, such applications will do our collegiate teaching a lot of good, both to students and teachers. The The National Council of Teachers of Mathematics devoted a recent yearbook to practical applications of mathematics. It would be splendid if we could have a running show of such applications. It would on the other hand, do us good to be challenged by the practical engineer to supply him with methods more adaptable to his needs. Mathematics of necessity dwells in an ivory tower. But it can only afford to do so as long as it stands ready to go out and do any kind of a chore that falls within its domain, whenever such a call is issued to it.

"What I am more worried about is where the contributions will come from. They must come from the engineer first. It is he who must first tell what his troubles are. It is also to the engineer to give what annswers to the questions are available at present. Then you need an editor. I presume Mr. Reese may be willing to take the job, or recommend somebody for it. The other side of the medal is how can we reach the engineer after we have the information he wants. All these questions will find their answers as we go along. The main point is to get started."

S. T. SANDERS.

Certain Expressions Related to Fourier Series*

By J. M. Dobbie
Northwestern University

1. *Introduction.* Recently Temple[1] and Simmons[2] discussed the equations

$$u^{(n)}(x) \pm k^n u(x) = 0, \quad k > 0,$$

with special attention to the case $k=1$. The purpose of this paper is to point out some important connections between such equations and Fourier series expansions. The paper is largely expository, although the author has not seen in print most of section 3.

Camp[3] showed that the expansion problem associated with the differential system

$$u'(x) + \lambda u(x) = 0,$$

$$u(\pi) - u(-\pi) = 0,$$

gives rise to Fourier series on the interval $(-\pi, \pi)$. By constructing the Green's function,[5] he was able to evaluate the resulting contour integral to obtain the convergence theorem for the expansion. He applied the results to multiple Fourier series expansions[3] as well as to multiple Birkhoff series expansions.[4] Carman[6] solved the corresponding problem for the system

$$u''(x) + \lambda u(x) = 0,$$

$$u'(\pi) - u'(-\pi) = 0,$$

$$u(\pi) - u(-\pi) = 0.$$

Much earlier Birkhoff[7] had discussed the equation

$$u^{(n)}(x) + * + p_2(x)u^{(n-2)}(x) + \cdots + (p_n(x) + \lambda)u(x) = 0,$$

$p_i(x)(i = 2,3, \cdots, n)$ having continuous derivatives of all orders on the closed interval considered, together with n linear boundary conditions which are regular.† In comparing these series of Birkhoff with Fourier

*Presented to the Society November 26, 1938 under the title *A note on a convergence proof for Fourier series.*

†For the definition of regularity see pp. 382, 383 of Birkhoff's paper.

series, Stone[8] showed that the expansion problem associated with the differential system

$$u^{(n)}(x)+\lambda u(x)=0, \tag{1}$$

$$u^{(r)}(\pi)-u^{(r)}(-\pi)=0, \quad r=0,1,2,\cdots,n-1, \tag{2}$$

gives rise to Fourier series by using fairly well known properties of the Green's function.

In this paper it is observed that the explicit determination of the Green's function shows that the system (1)(2) gives rise to Fourier series and enables us to express the Fourier partial sums in terms of the Green's function. It is unnecessary to divide the analysis into odd and even cases. Finally, we apply the results to multiple Fourier series expansions.

2. *The formal expansion theory.** We present the usual formal expansion theory in order to show how it is connected with the work of the next section.

It is well known that a necessary and sufficient condition for a non-trivial solution of the system (1)(2) is that $\lambda=-m^n i^n$, where m is an integer and $i=\sqrt{-1}$. These values are called characteristic values. If $\lambda=\rho^n$, the general solution of (1) is

$$u(x)=\sum_{j=1}^{n} c_j e^{\rho w_j x},$$

where the w_j are the n nth roots of -1, and the characteristic values for ρ becomes $\rho_j=mi/w_j$, $(j=1,2,3,\cdots,n;\ m=0,\ \pm1,\ \pm2,\ \pm3,\cdots)$.

With the system (1)(2) we associate the adjoint system

$$(-1)^n v^{(n)}(x)+\lambda v(x)=0, \tag{3}$$

$$v^{(r)}(\pi)-v^{(r)}(-\pi)=0, \quad r=0,1,2,\cdots,n-1. \tag{4}$$

The characteristic values for this system are the same as those for the system (1)(2). Let u_j and v_j be corresponding solutions of the systems (1)(2) and (3)(4) for the same characteristic value ρ_j. Then, it is easy to show that

$$\int_{-\pi}^{\pi} u_j v_k \, dx=0, \quad (j\neq k),$$

*See Birkhoff, *loc cit.*, for a more general treatment.

the biorthogonality condition. The formal expansion for $f(x)$ in terms of solutions of (1)(2) is

$$f(x) \sim \sum_{j=1}^{\infty} \left[\frac{\int_{-\pi}^{\pi} f(x) v_j(x) dx}{\int_{-\pi}^{\pi} u_j(x) v_j(x) dx} \right] u_j(x). \tag{5}$$

When ρ is not a characteristic value of the system (1)(2) there exists a unique function $\overline{G}(x,y;\ \rho)$ such that the sum of the first k terms in the expansion (5) can be written as

$$\frac{1}{2\pi i} \int_{C_k} \int_{-\pi}^{\pi} \overline{G}(x,y;\ \rho) f(y) dy\, d\rho,$$

where C_k is a circle with center at the origin and radius r_k, $k < r_k < k+1$. Moreover, if we consider the system

$$w^{(n)}(x) + \lambda w(x) = f(x),$$

$$w^{(r)}(\pi) - w^{(r)}(-\pi) = 0, \quad r = 0,1,2,3,\cdots, n-1,$$

the solution is given by $w(x) = \int_{-\pi}^{\pi} \overline{G}(x,y;\ \rho) f(y) dy.$

In the next section we construct the Green's function and show how the partial sums of the expansion are related to the partial sums of the Fourier series.

3. *Expansions in terms of Green's functions. Fourier series.* We seek a function $G(x,y;\ \rho)$ which, for ρ not equal to a characteristic value, satisfies the boundary conditions (2) and also satisfies equation (1) at every point of the interval $(-\pi, \pi)$ except at an interior point y at which point $G, G', G'', \cdots, G^{(n-2)}$ are continuous but $G^{(n-1)}$ has a jump discontinuity of 1. If such a function exists, it must have the form

$$G(x,y;\ \rho) = \begin{cases} \sum_{j=1}^{n} a_j e^{w_j \rho x}, & -\pi \leq x < y; \\ \sum_{j=1}^{n} b_j e^{w_j \rho x}, & y < x \leq \pi. \end{cases}$$

We get n linear equations in a_j and b_j from the boundary conditions, $n-1$ more from the continuity at $x=y$ of G and its first $n-2$ derivatives, and one from the jump discontinuity at $x=y$ of the $n-1$st derivative:

$$\begin{cases} \sum_{j=1}^{n} w_j^r(b_j e^{w_j\rho\pi} - a_j e^{-w_j\rho\pi}) = 0, \quad (r=0,1,2,\cdots, n-1); \\ \sum_{j=1}^{n} w_j^r e^{w_j\rho y}(b_j - a_j) = 0, \quad (r=0,1,2,\cdots, n-2); \\ \rho^{n-1}\sum_{j=1}^{n} w_j^{n-1} e^{w_j\rho y}(b_j - a_j) = 1. \end{cases}$$

These equations can be solved easily to obtain the Green's function

$$G(x,y;\ \rho) = \frac{1}{n\rho^{n-1}} \sum_{j=1}^{n} \frac{w_j e^{w_j\rho[x-y+\pi \operatorname{sgn}(y-x)]}}{e^{w_j\rho\pi} - e^{-w_j\rho\pi}}, \tag{6}$$

in which $\operatorname{sgn}(y-x)$ means the sign of $(y-x)$. The reader can verify that $\rho^{n-1}G$ plays the role of $\overline{G}$ in the formal expansion theory.

We now show that the formal expansion is a Fourier series. It is evident that $\rho^{n-1}G(x,y;\ \rho)$ has simple poles at the characteristic values $\rho_j = mi/w_j$ and a direct computation gives the residues $R_{m,j}(x,y)$ at these points as

$$R_{m,j}(x,y) = R_m(x,y) = \frac{1}{n} \cdot \frac{1}{2\pi} e^{mi(x-y)}, \quad m=0,\ \pm 1,\ \pm 2, \cdots.$$

Then $$R_m(x,y) + R_{-m}(x,y) = \frac{1}{n\pi}\cos(mx - my),$$ and

$$\frac{1}{2\pi i}\int_{C_m - C_{m-1}} \rho^{n-1}G(x,y;\ \rho)d\rho = \frac{1}{\pi}(\cos mx \cos my + \sin mx \sin my),$$

where C_m is a circle with center at the origin and radius r_m, $m < r_m < m+1$, it being understood that when $m=0$, $C_m - C_{m-1}$ is to be replaced by C_0. The general term in the Fourier series expansion of $f(x)$ is

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (\cos mx \cos my + \sin mx \sin my) f(y)dy$$
$$= \frac{1}{2\pi i}\int_{C_m - C_{m-1}}\int_{-\pi}^{\pi} \rho^{n-1}G(x,y;\ \rho)f(y)dy\, d\rho.$$

Hence, the sum of the constant and the next k terms of the expansion becomes

$$S_k(x) = \frac{1}{2\pi i}\int_{C_k}\int_{-\pi}^{\pi} \rho^{n-1}G(x,y;\,\rho)f(y)dy\,d\rho. \tag{7}$$

This equation shows that the expansion is a Fourier series and expresses the partial sums in terms of the Green's function.

Incidentally, we can get the well known convergence theorem. By equation (6) we can write (7) as

$$S_k(x) = \frac{1}{2\pi i n}\int_{C_k}\int_{-\pi}^{\pi} \sum_{j=1}^{n} \frac{w_j e^{w_j\rho[x-y+\pi \operatorname{sgn}(y-x)]}}{e^{w_j\rho\pi}-e^{-w_j\rho\pi}} f(y)dy\,d\rho.$$

Now make the substitution $\sigma_j = w_j\rho$, and $S_k(x)$ becomes

$$S_k(x) = \frac{1}{n}\sum_{j=1}^{n} \frac{1}{2\pi i}\int_{C_k}\int_{-\pi}^{\pi} \frac{e^{\sigma_j[x-y+\pi \operatorname{sgn}(y-x)]}}{e^{\sigma_j\pi}-e^{-\sigma_j\pi}} f(y)dy\,d\sigma_j.$$

$$\lim_{k\to\infty} S_k(x) = \frac{1}{n}\sum_{j=1}^{n} \lim_{k\to\infty}\frac{1}{2\pi i}\int_{C_k}\int_{-\pi}^{\pi} \frac{e^{\sigma_j[x-y+\pi \operatorname{sgn}(y-x)]}}{e^{\sigma_j\pi}-e^{-\sigma_j\pi}} f(y)dy\,d\sigma_j$$

$$= \frac{1}{n}\cdot n\cdot \tfrac{1}{2}[f(x+0)+f(x-0)] = \tfrac{1}{2}[f(x+0)+f(x-0)],$$

the limit being that evaluated by Camp for the case $n=1$. It is easy to show that the expansion converges to $\frac{1}{2}[f(-\pi+0)+f(\pi-0)]$ if $x=\pm\pi$.

4. *Multiple Fourier series expansions.* Let $f(x_1,x_2,\cdots,x_s)$ be made up, in the region $-\pi\le x_j\le\pi$, $(j=1,2,3,\cdots,s)$, of a finite number of pieces each of which is continuous in the region and has continuous first partial derivatives with respect to $x_1,x_2,\cdots,x_s$. The s-fold Fourier series for $f(x_1,x_2,\cdots,x_s)$ is

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_s=0}^{\infty} h_{k_1k_2\cdots k_s}\sum_{r=1}^{2^{s(r)}} A_{k_1k_2\cdots k_s}\cdot\prod_{j=1}^{s}{}^{(r)}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(k_jx_j),$$

where $h_{k_1k_2\cdots k_s}=1/2^p$, p being the number of k's which are zero; and the last sum is taken over the 2^s choices of a cosine or sine factor in the product. The coefficients are

$$A^{(r)}{}_{k_1k_2\cdots k_s}$$
$$=\frac{1}{\pi^s}\int_{-\pi}^{\pi}\int_{\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\prod_{j=1}^{s}{}^{(r)}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(k_jy_j)f(y_1,y_2,\ldots,y_s)dy_1dy_2\cdots dy_s.$$

Hence, the general term of the s-fold series is

$$h_{k_1k_2\cdots k_s}\cdot\frac{1}{\pi^s}\sum_{r=1}^{2^s}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\prod_{j=1}^{s}{}^{(r)}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(k_jx_j)$$

$$\cdot\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(k_jy_j)f(y_1,y_2,\cdots y_s)dy_1dy_2\cdots dy_s$$

$$=h_{k_1k_2\cdots k_s}$$

$$\cdot\frac{1}{\pi^s}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\prod_{j=1}^{s}\cos k_j(x_j-y_j)f(y_1,y_2,\cdots y_s)dy_1dy_2\cdots dy_s.$$

By equation (5) and the equation just preceding it, the partial sum is

$$S_{r_1r_2\cdots r_s}\equiv\sum_{k_1=0}^{r_1}\sum_{k_2=0}^{r_2}\cdots\sum_{k_s=0}^{r_s}h_{k_1k_2\cdots k_s}\sum_{r=1}^{2^s}A^{(r)}_{k_1k_2\cdots k_s}\cdot\prod_{j=1}^{s}{}^{(r)}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(k_jx_j)$$

$$=\frac{1}{(2\pi i)^s}\int_{C_{r_1}}\int_{-\pi}^{\pi}\rho_1^{n-1}G_1\int_{C_{r_2}}\int_{-\pi}^{\pi}\rho_2^{n-1}G_2\cdots$$

$$\int_{C_{r_s}}\int_{-\pi}^{\pi}\rho_s^{n-1}G_sf(y_1,y_2,\cdots,y_s)dy_1d\rho_1dy_2d\rho_2\cdots dy_sd\rho_s,$$

where $G_j\ (j=1,2,\cdots,s)$ is the Green's function $G(x_j,y_j;\rho_j)$ for $f(x_1,x_2,\cdots,x_s)$ considered as a function of x_j, and C_{r_j} is a circle of radius $r_j+(1/2)$ with center at the origin in the ρ_j-plane.

From the result of Camp[3] and section 3

$$\frac{1}{2\pi i}\int_{C_{r_s}}\int_{-\pi}^{\pi}\rho_s^{n-1}G_sf(y_1,y_2,\cdots,y_s)dy_sd\rho_s$$

converges to

$$\tfrac{1}{2}[f(y_1,y_2,\cdots,y_{s-1},x_s-0)+f(y_1,y_2,\cdots,y_{s-1},x_s+0)],$$

the convergence being uniform with respect to $y_1,y_2,\cdots y_{s-1}$. Hence, we can repeat the argument to show that the s-fold Fourier series converges to

$$\frac{1}{2^s}\sum[f(x_1\pm 0,\ x_2\pm 0,\cdots,\ x_s\pm 0)],$$

$x_j\pm 0$ being replaced by $-\pi+0$ and $\pi-0$ when $x_j=\pm\pi$.

5. *Partial differential systems.* We now can treat the partial differential equation

$$\sum_{j=1}^{s} \frac{\partial^{(n)} u}{\partial x_j^{(n)}} + \lambda u = 0,$$

with suitable boundary conditions. The treatment is so much like that of Camp[3] for the case $n=1$ that the writer does not think it necessary to display the details.

REFERENCES

[1] V. B. Temple, NATIONAL MATHEMATICS MAGAZINE, March, 1939, pp. 263-271;

[2] H. A. Simmons, NATIONAL MATHEMATICS MAGAZINE, May, 1940, pp. 1-4.

[3] C. C. Camp, Trans. of A. M. S., Vol. 25 (1923), pp. 123-134.

[4] C. C. Camp, Trans. of A. M. S., Vol. 25 (1923), pp. 338-342.

[5] Maxime Bocher, Bulletin of A. M. S., Vol. 7 (1901), pp. 297-299.

[6] M. G. Carman, Bulletin of A. M. S., Vol. 30 (1924), pp. 410-416.

[7] G. D. Birkhoff, Trans. of A. M. S., Vol. 9 (1908), pp. 373-395.

[8] M. H. Stone, Trans. of A. M. S., Vol. 28(1926), pp. 695-761; especially pp. 709-712.

An Infinite Series

By William E. Byrne
Virginia Military Institute

In *Solutions de Questions de Mathématiques Spéciales, Première partie, Algèbre, Analyse, Mécanique* par A. Tétrel (cinquième edition, Librairie Croville, Paris, no date), there appears the following problem, Number 134, page 37:

"Sum the series $$u_n = \frac{1}{1^2+2^2+\cdots+n^2}.$$

It is known that $$1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Consequently $$u_n = \frac{6}{n(n+1)(2n+1)}. = \frac{24}{(2n)(2n+1)(2n+2)}$$

Since $n^3 u_n$ tends toward 3, we conclude that the series is convergent. Splitting u_n into simple elements, we find

$$u_n = \frac{6}{n} + \frac{6}{n+1} - \frac{24}{2n+1}.$$

Consequently,

$$u_1 = \frac{6}{1} + \frac{6}{2} - \frac{24}{3},$$

$$u_2 = \frac{6}{2} + \frac{6}{3} - \frac{24}{5},$$

$$u_3 = \frac{6}{3} + \frac{6}{4} - \frac{24}{7}, \cdots$$

$$S = \frac{6}{1} + 2\cdot\frac{6}{2} - 2\cdot\frac{6}{3} + 2\cdot\frac{6}{4} - 2\cdot\frac{6}{5} + \cdots$$

$$= 6 + 12\left[\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots\right].$$

*Published in Brussels, 1874-1880 under the editorship of E. Catalan (1814-1894).

If we recall that the alternating harmonic series has log 2 as its sum, we see that $S=6+12(1-\log 2)=18-12\log 2$."

Although the solution given above is incorrect, it may serve to illustrate the necessity of due care in manipulating non-absolutely convergent series. The error arose from removing parentheses in a series, thus replacing an absolutely convergent series by three conditionally convergent series, and then making unwarranted rearrangements.

This simple solution was suggested by Professor T. L. Smith of Carnegie Institute of Technology:

$$u_1=\frac{6}{1}+\frac{6}{2}-\frac{24}{3}$$

$$u_2=\frac{6}{2}+\frac{6}{3}-\frac{24}{5}$$

$$u_3=\frac{6}{3}+\frac{6}{4}-\frac{24}{7}$$

$$u_4=\frac{6}{4}+\frac{6}{5}-\frac{24}{9}$$

$$\cdots$$

$$u_n=\frac{6}{n}+\frac{6}{n+1}-\frac{24}{2n+1}$$

$$u_{n+1}=\frac{6}{n+1}+\cdots$$

$$S_n=6-\frac{24}{3}+\left(\frac{6}{2}+\frac{6}{2}\right)-\frac{24}{5}+\left(\frac{6}{3}+\frac{6}{3}\right)$$

$$-\frac{24}{7}+\left(\frac{6}{4}+\frac{6}{4}\right)-\frac{24}{9}+\cdots$$

$$+\left(\frac{6}{n}+\frac{6}{n}\right)-\frac{24}{2n+1}+\frac{6}{n+1}$$

$$=6-\frac{24}{3}+\frac{24}{4}-\frac{24}{5}+\cdots-\frac{24}{2n+1}+\frac{6}{n+1}$$

$$=6-24\left(\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{2n+1}\right)+\frac{6}{n+1}$$

$$=6+12-24\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n+1}\right)+\frac{6}{n+1}$$

$$S=\lim_{n\to\infty} S_n=18-24\log 2.$$

The writer of this article had previously solved the problem from an entirely different point of view, which has some general interest.

Consider $$f(x)=6\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)(2n+1)}$$

which is absolutely convergent for $-1\leqq x\leqq 1$. We have

$$f(1)=\lim_{x\to 1^-} f(x).$$

If $$\alpha_1(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}, \qquad \alpha_1'(x)=\frac{1}{1-x}$$

$$\alpha_1(x)=-\log(1-x), \quad |x|<|$$

Likewise, if $$\alpha_2(x)=\sum_{n=1}^{\infty}\frac{x^n}{n+1}, \qquad x\alpha_2(x)=\alpha_1(x)-x$$

$$=-\log(1-x)-x$$

$$\alpha_2(x)=-\frac{1}{x}\log(1-x)-1, \quad 0<|x|<|.$$

And if $$\alpha_3(x)=\sum_{n=1}^{\infty}\frac{x^n}{2n+1},$$

the transformation $x=t^2$, $t=\sqrt{x}$ gives

$$\alpha_3(t^2)=\frac{t^2}{3}+\frac{t^4}{5}+\cdots+\frac{t^{2n}}{2n+1}+\cdots$$

$$\frac{d}{dt}[t\alpha_3(t^2)]=t^2+t^4+\cdots+t^{2n}+\cdots=\frac{1}{1-t^2}-1$$

$$t\alpha_3(t^2)=\frac{1}{2}\log\frac{1+t}{1-t}-t$$

$$\alpha_3(x)=\frac{1}{2\sqrt{x}}\log\frac{1+\sqrt{x}}{1-\sqrt{x}}-1, \qquad 0<x<1.$$

Hence $$f(x)=18-6\left[\left(1+\frac{1}{x}\right)\log(1-\sqrt{x})+\left(1+\frac{1}{x}\right)\log(1+\sqrt{x})\right.$$
$$\left.+\frac{2}{\sqrt{x}}\log(1+\sqrt{x})-\frac{2}{\sqrt{x}}\log(1-\sqrt{x})\right]$$
$$=18-\frac{6}{x}(1-\sqrt{x})^2\log(1-\sqrt{x})-\frac{6}{x}(1+\sqrt{x})^2\log(1+\sqrt{x})$$

$$\lim_{x\to 1^-} f(x)=18-24\log 2.$$

Humanism and History of Mathematics

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The Influence of Mathematics on the Philosophy of Descartes

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1. *Introduction.* The history of mathematics and the history of philosophy might be thought of as surfaces which intersect in the lives of many men. Of the many men who have been famous as mathematicians and as philosophers, René Descartes was perhaps the most outstanding. Living in the first half of the seventeenth century, he was the father of analytic geometry and of modern philosophy. Let us try to see whether mathematics influenced his philosophy and to determine the nature of such influence if it existed.

At the time of Descartes the term "philosophy" had a much wider meaning than it has today. Philosophy included all there was of what we call "science" while the term "science" simply connoted any branch of knowledge. Descartes made no distinction whatever between science and philosophy, declaring:

> the word Philosophy signifies the study of wisdom, and by wisdom is to be understood not merely prudence in the management of affairs, but a perfect knowledge of all that man can know, as well for the conduct of his life as for the preservation of his health and the invention of all the arts.[1]

He used the term "philosophy" in two senses. In its widest meaning philosophy included metaphysics and physics. Thus he declared that philosophy was like a tree:

> all philosophy is like a tree, of which the roots are Metaphysics, the trunk is Physics, and the branches which go out from this trunk are all other sciences, which reduce to three principles, namely: Medicine, Mechanics, and Morals....[2]

[1] René Descartes, *Les Principes de la Philosophie, Ecrits en Latin, et Traduits en François par un de ses Amis* (Paris: Henry Le Gras et Edme Pepingue, 1651), preface.
[2] *Ibid.*

In its narrower meaning "philosophy" was synonymous with natural philosophy. According to Keeling:

> Evidently, then, what Descartes calls "philosophy" is not what we today refer to by that name. In its narrower sense it includes all the physical doctrines presented in the *Monde*, and these are certainly not philosophy nor metaphysics in the sense these words have been employed during the last century and a half.[3]

Descartes made it clear that he considered his work in optics and meterology to be parts of philosophy. Thus he declared in his letter to Dinet:

> in this small number of meditations which I published, all the principles of the philosophy which I am preparing are contained; and in the Dioptric and Meteors I have deduced from these principles many particular things which show what is the manner of my reasoning; and that is why, although I am not yet setting forth all that philosophy, I yet consider that what I have already given forth suffices to make known what it will be.[4]

The study of the influence of mathematics on the philosophy of Descartes will include the study of the influence of mathematics on the science with which he dealt. At an earlier period there was no science as we understand the meaning of the term today. It was precisely at the time of Descartes that modern science began to develop. Thus, to show the influence of mathematics on science after the time of Descartes would not show its influence on philosophy.

2. *Evidence in General.* Let us consider some evidence to show that mathematics influenced the philosophy of Descartes. Many authorities have asserted that mathematics was related to his philosophy. Thus Turnbull declared that

> from the beginning of the seventeenth century the number [of mathematicians] increased .... rapidly .... In France there were as many mathematicians of genius as Europe had produced during the preceding millenium.... The age was mathematical; the habits of mind were mathematical; and its methods were deemed necessary for an exact philosophy, or an exact anything else. It was the era when what is called modern philosophy began; and the pioneers among its philosophers, like the Greek philosophers of old, were expert mathematicians. They were Descartes and Leibniz.[5]

Rideau declared:

> Descartes was a mathematician and with so much fervor that he found in mathematical deduction not only the type and the key of all truth, but also the most proper and easy exercise of intellectual culture.

[3] S. V. Keeling, *Descartes* (London: Ernest Benn, Limited, 1934), p. 59.

[4] E. S. Haldane and G. R. T. Ross, *The Philosophical Works of Descartes Rendered into English* (Cambridge: The University Press, 1912), II, 375.

[5] H. W. Turnbull, *The Great Mathematicians* (London: Methuen and Company, Limited, 1929), p. 70.

> Only mathematics, by its rigor, its fruitfulness ("fécondité"), its generality, gives to the mind a perfect satisfaction, an unmixed joy.[6]

According to Burtt:

> The work of Descartes had an enormous influence throughout all Europe during the latter part of the seventeenth century, largely because he was not only a great mathematician and anatomist, but also a powerful philosophical genius, who treated afresh, and with a remarkably catholic reach, all the big problems of the age by hitching them up in one fashion or another to the chariot of victorious mathematical science.[7]

Baumann declared that in Descartes' major doctrines

> there rules in every domain a logical and mathematical, exactly expressed, spatial and geometrical, indeed an arithmetical and temporal, fundamental principle which influenced his thinking ("Gedankbildungen") positively or negatively; ....[8]

According to Aster:

> Descartes is correctly designated as the "Father of Modern Philosophy," for with him begins the characteristic manner of thinking of the seventeenth and eighteenth centuries, which is so plainly contrasted with the manner of thinking of the Renaissance Philosophy.... In contrast with Bruno, Descartes thinks in the manner of natural science and mathematics ("naturwissenschaftlich-mathematisch"), in which mathematical thought wins an authoritative influence on philosophy. It is his goal with which the subsequent philosophers were especially occupied: metaphysical knowledge embracing the entire world by the mathematical method....[9]

Finally, Tabulski declared: "Thus Descartes, directed by mathematics, set up the principle and the method of speculative philsophy, from which all of his successors up until Kant did not deviate."[10]

Many writers have attested to Descartes' synthesis of mathematics and philosophy, but no one has analyzed his entire philosophy in relation to mathematics. Let us turn now to the consideration of the effect of mathematics on the various phases of the philosophy of Descartes. For this purpose, philosophy may be divided into the following problems: the problem of method, the problem of episte-

[6] Émile Rideau, *Descartes, Pascal, Bergson* (Paris: Boivin et Cie., 1937), p. 12.

[7] E. A. Burtt, *The Metaphysical Foundations of Modern Physical Science* (New York: Harcourt Brace and Company, 1932), p. 117.

[8] J. J. Baumann, *Die Lehren von Raum, Zeit, und Mathematik in der Neueren Philosophie* (Berlin: Georg Reimer, 1868), I, 156.

[9] Ernest von Aster, *Einführung in die Philosophie, Descartes* (München: Rosl und Cie., 1921), pp. 21, 22.

[10] August Tabulski, *Ueber den Einfluss der Mathematik auf die Geschictliche, Entwickelung der Philosophie bis auf Kant*, Dissertation, University of Jena (Leipzig: Paul Rhode, 1868), p. 25.

mology, the problem of metaphysics, the problem of natural philosophy, and the problems of practical philosophy.

3. *Method.* The method of a philosopher is important, for it is the point of departure of his system. When Descartes was only twenty-three or twenty-four years of age, he began to think of mathematics in a generalized way. This became the concept of *mathesis* or *universal mathematics* which included the entire scope of *order* and *measure* and of which ordinary mathematics was the husk rather than the core, the envelope rather than the contents. Keeling pointed out the relation of Descartes' *mathesis* to modern concepts:

> Descartes certainly reached this conception, but he did not, like Leibniz, work it out in any considerable detail. Examination of the structure of ordinary algebra (which Descartes would have agreed with Leibniz is 'nothing else than the characteristic of indeterminate numbers or qualities') had suggested the idea of constructing another—a 'Universal' Characteristic—of which the former would be simply a special and restricted application. Hence the enterprise (which in our own day has reached a much fuller development in the 'logistic' or mathematical logic of Peano, Whitehead and Russell) of creating a 'characteristic' applicable to any possible object of thought, a formal logic whose principles should be expressed in, and facilitated by, a convenient notation or symbolism.[11]

Descartes was, of course, the inventor of *analytic* geometry as contrasted with the synthetic or pure geometry of the Greeks. In philosophy, too, he considered analysis to be a more important technique than synthesis. But in philosophy his analysis was not the analysis of his mathematics, the expressing of problems in the form of algebraic equations. Rather it was the analysis or decomposition which had been invented by Plato and used by the Greek geometers. Descartes promised to extend the solution by mathematical analysis to problems of any type whatever, but no tangible evidence of such efforts has ever been found. Only in the twentieth century has symbolic logic extended mathematical analysis to all types of propositions. This has been the work of Whitehead, Russell, and others. One might be inclined to think that the same techniques are used in philosophy, but only much later than they are used in mathematics.

The technique of enumeration or induction which Descartes used in his philosophical method was closely akin to the mathematical process of finding all possible combinations. Snow declared that it was by the understanding of mathematical method, which implies induction as well as deduction, that Descartes developed his method.[12]

[11] *Op. cit.*, p. 45.

[12] A. J. Snow, "Descartes' Method and the Revival of Interest in Mathematics," *The Monist*, XXXIII (1923), 615.

In no phase of Descartes' method is the synthesis of mathematics and philosophy more evident than in deduction. He abandoned the deduction of the syllogism and patterned his deduction after that of mathematics. The syllogism could put known truths in order but it could not produce knowledge. The logic of deduction that was of practical value was the method of mathematics. Just as mathematicians deduce theorems from an array of postulates and assumptions, Descartes used deduction from a set of first principles, proceeding always from the simplest to the most complicated. According to Trognitz, Descartes found that deduction in his philosophical investigations corresponded to the deductive character of mathematics:

> After we understand the idea of deduction, we are confronted by the question: "What are the conditions which are required by deduction?" We obtain these conditions if we fasten our attention on deduction from the standpoint of mathematics.... The philosophical investigation proceeds in an entirely similar way.[13]

Descartes considered *order* one of the fundamental ideas of mathematics. The objects of mathematics could be placed in series, of which all the parts were known by means of one another. In every such series there was a dominant element, which was the standard upon which the rest of the series depended, and hence "absolute" with regard to the series. In exactly the same way he sought the absolutes in whatever matter he chose to investigate by means of his method.

4. *Epistemology.* Epistemology, the problem of knowledge, is closely related to method, since a philosopher's method is for the purpose of obtaining knowledge. Descartes declared that the main reason for regarding his views as superior to those of any other was the fact that he had adopted a kind of philosophizing which cancelled out as invalid any reason that was not mathematical and clear.[14] Trognitz declared that by means of mathematics Descartes first located the true path to knowledge. As a product in algebra was the resultant of its separate factors, so Truth was obtained by putting together elementary truths.[15] Just as in the realm of mathematics he could march confidently to his goal by means of analysis, so could he obtain universal knowledge by means of a sound method.

[13] B. Trognitz, "Die Mathematische Methode in Descartes' Philosophischem Systeme," *Programm des Realgymnasiums zu Saalfeld* (Saalfeld: Wiedemannsche Hofbuchdruckerei, 1887), p. 6.

[14] Victor Cousin, *Oeuvres de Descartes* (Paris: F. G. Levrault, 1824-1826), VII, 121, 434; VIII, 123.

[15] *Op. cit.*, p. 5.

In epistemology Descartes' universal mathematics was the theoretical extension of order and measure to any type of knowledge whatever. Mathematics appeared to be the only hope of finding an entering wedge to the problem of getting exact knowledge. Compared with mathematics all the other sciences were merely exercises of the memory or abstract dialectics and were incapable of supplying irrefutable conclusions. Descartes declared in Part I of the *Discours:*

> I was especially pleased with mathematics, on account of the certitude and evidence of its reasons; but I did not notice at all its true usage, and considering only the mechanical arts, I was surprised that foundations so strong and solid had not been used for larger buildings.[16]

His procedure was to admit no idea which was not as clear and distinct as the foundations of mathematics. "Clear and distinct" ideas were not only best illustrated but were also epitomized by mathematics. Descartes sought to find by means of analysis a clear and distinct idea to use as the foundation of knowledge. Just as a mathematician analyzes a problem to find a datum to which are related all the variables of the problem, Descartes obtained the *cogito ergo sum* ("I think, therefore I am") as the product of his analysis. The *cogito* was an axiomatic statement, a statement of identity. It might be thought of as the fulcrum or solid point of balance of the system of Descartes.

Thus he obtained true knowledge by making it a necessary and sufficient condition that everything which could be perceived as clearly and distinctly as the truths of mathematics was also true. His success in this attempt was appraised by the author of the letter which introduced Descartes' *Passions of the Soul.* It was first translated into English in the following quaint way:

> the certainty already discovered in the Mathematicks makes much for you: for it is a science wherein you are acknowledged to be so excellent,.... and it is seen by what you have published concerning Geometry, that you there so determine how far humane capacity can reach, and which is the way of solving every Manner of Scruple, that it seemes you have reached the whole harvest, whereof those who write before you have onely cropped some ears; and your successors can be but gleaners,....[17]

5. *Metaphysics.* Metaphysics, the problem of ultimate reality, is considered by many men to be the most important problem in philosophy. Descartes was a dualist, holding that there were two

[16] René Descartes, *Discours de la Methode, pour bien Conduire sa Raison, & Chercher la Verité dans les Sciences; plus la Dioptrique, les Meteores, et la Geometrie qui sont des Essais de cete Methode* (Leyden: Jan Maire, 1637), p. 9.

[17] René Descartes, *The Passions of the Soule; translated out of French into English* (London: J. Martin and J. Ridley, 1650), preface.

ultimate realities: mind and matter, both of which showed the existence of a supreme being. Let us see how mathematics was related to his treatment of the problem of metaphysics. In a letter to Father Mersenne, his best friend, Descartes declared:

> I think I have found the way in which metaphysical truths can be demonstrated in a fashion which is more evident than the demonstrations of geometry: I say that according to my own judgment, for I do not know whether I will be able to persuade others.[18]

Descartes thought that the attribute of matter or extended substance was mathematical extension. On the other hand, mind or thinking substance could not be mathematical since it was the antitheiss of mathematical extension.

The principal effect of mathematics on the metaphysics of Descartes was in connection with his treatment of the problem of God. In the second book which he published, he answered various objections to his metaphysics. The second "Objections" were ended with the following request:

> There are, Monsieur, the difficulties upon which we wish that you would shed a greater light, in order that the reading of your very subtle and true "Meditations" be profitable to everybody. That is why it would be a very useful thing, if, at the end of your solutions, after having first advanced some definitions, postulates, and axioms, you conclude everything according to the method of geometers, in which you are so well versed, in order that your readers may be satisfied at a glance, and in order that you fill their minds with knowledge of the divinity.[19]

In his "Reply" Descartes declared that he noticed two things in the manner of writing of mathematicians: order and method of proof. He declared that he used mathematical order in his "Metaphysical Meditations." His method of demonstration was analysis, similar to mathematical analysis, rather than synthesis. In accordance with the request, however, he reluctantly agreed to display some of his metaphysics in the synthetic geometrical form of Euclid's *Elements*. The result was Descartes' "Reasons which Prove the Existence of God and the Difference between the Mind and Body of Man, Displayed in Geometrical Fashion." There were ten definitions, seven postulates, ten axioms or common notions, four propositions, and one corollary.

Some men concluded that the deductive synthetic expression of Descartes metaphysics was the significant use of mathematics in his philosophy. This became the almost worthless *more geometrico* of the

[18] Claude Clerselier, *Lettres de M. Descartes, Touchant la Morale, la Physique, la Medicine, Nouvelle Edition* (Paris: Charles Angot, 1667), II, 478.

[19] René Descartes, *Les Meditations Metaphysique De René Descartes Touchant la Premier Philosophie* (Paris: Jean Camvsat et Pierre Le Petit, 1647), p. 165.

eighteenth century. Some writers, seeing that the *more geometrico* was of little value, have discounted the importance of mathematics in the philosophy of Descartes. They have overlooked the fact that he used analysis, similar to mathematical analysis, as the essential technique in forming his metaphysics.

Descartes wished to admit nothing into his metaphysics which did not have certainty equal to that of mathematics. On the other hand, he could not be absolutely certain of the validity of mathematics until he knew that God existed. In Meditation V he declared:

> Thus, for example, when I consider the nature of the triangle, I, who am a little versed in geometry, know that the three angles are equal to two right angles, and I find it impossible to believe otherwise, while I apply myself to the demonstration; but as soon as I cease attending to the process of proof, although I still remember that I had a clear comprehension of it, yet I may readily come to doubt the truth of it, if I ignore the fact that there is a God.[20]

To base the concept of God upon the certainty of mathematics and the certainty of mathematics upon the existence of God seems to be reasoning in a circle; but if the circle is large enough, it is not a vicious one. Thus mathematics is based on assumptions and the more propositions a mathematician deduces from these assumptions, the surer he becomes of their validity, since he returns to them time after time.

Descartes gave three proofs of the existence of God which he declared to be similar to the proofs of mathematicians. He asserted that his ontological proof was parallel to a proof of geometry. In answering the objections of Gassendi, he declared:

> You are plainly in error when you say that the existence of God is not demonstrated in the same way that is demonstrated that the three angles of a triangle are equal to two right angles; for the reason is parallel in both of them; except that the demonstration which proves the existence of God is much simpler and more evident than the other.[21]

An early English critic showed how little he understood the method of Descartes when he criticized the ontological proof in the following way:

> And thus he again supposes "That all Men must be so far Geometrically knowing, or they will be deficient, or without demonstrative Conviction that the Deity does Exist."
>
> If this doctrine were true, it were no less requisite that all Mankind should have Recourse, betimes, to the School of "Euclid"; where they might be instructed.... Which were very disconsonant to his Idea of any Perfection in a Triangle, as he would parallel it, to the proving of the Existence of the Deity; ....[22]

[20] *Meditations*, (1647), pp. 84, 85.

[21] *Ibid.*, p. 583.

[22] Edward Howard of Berks, *Remarks on the New Philosophy of Descartes* (London: T. Ballard, 1701), pp. 18, 19.

6. *Natural Philosophy.* Natural philosophy considers the problem of the external world. For Descartes the external world included everything except his own mind (or soul or consciousness) and God. He regarded his body as being a machine governed by natural laws. Animals were nothing but machines, since they had no mind or soul. He had shown in his metaphysics that matter existed in so far as it agreed with the laws of abstract mathematics. The crux of the problem of natural philosophy was to determine how far the external world agreed with the objects of pure mathematics. From the beginning it was clear that Descartes wanted to extend the realm of pure geometry to the external world. He wrote to Mersenne on July 27, 1638:

> M. Des Argues is kind on account of the solicitation which he has for me because I do not wish to study geometry any more; but I have resolved to abandon only abstract geometry, that is, problems which serve only to exercise the mind, in order to have more leisure to cultivate another kind of geometry, which is intended for the problem of the explanation of the phenomena of nature. If he will take the trouble to consider what I have written about salt, snow, the rainbow, etc., he will indeed know that all my *physique* is nothing but geometry.[23]

Thus Descartes gave objective reality to geometry in the same way that Pythagoras, a mathematician who philosophized much earlier, gave objective reality to number. Descartes regarded all material bodies as being in reality mere modifications of geometrical extension. Color, taste, temperature, and other secondary qualities were merely translations or transformations of the mathematical essence of matter by the various human senses. He did not have the modern conception of mathematics as a postulational system of thought. To him mathematics was absolutely true and it would undoubtedly have surprised him to learn that there could be non-Euclidean geometry. For him the rules of mathematics were necessary laws, universal and absolute. According to Serrus, he "did not at all foresee the modern conception of mathematics as a hypothetical-deductive science,. . . ."[24]

Descartes' universal mathematics attained its highest development in natural philosophy. Order and measure were extended theoretically to all the problems of mechanics, music, physics, astronomy, biology, and optics.[25] Such a concept, the extension of the mathematical method to the universality of cosmological problems, was an innovation in the seventeenth century. The writer of the preface of

[23] Cousin, *Op. cit.*, VII, 121.

[24] Charles Serrus, *La Méthode de Descartes et son Application a la Métaphysique* (Paris: F. Alcan, 1933), p. 29.

[25] Leon Brunschvicg, *Les Étapes de la Philosophie Mathématique* (Paris: Felix Alcan, 1912), p. 113.

Descartes' *Treatise on Music* brought out the relation of mathematics to music. It was first translated into English thus:

> For, He must be.... An Arithmetician, to be able to explain the Causes of Motions Harmonical, by Numbers, and declare the mysteries of the new Algebraical Musick. A Geometrician; to evince, in great variety the Original of Intervalls Consono-dissonant, by the Geometrical, Algebraical, Mechanical Division of a Monochord.[26]

Descartes' physics was the most important development of his universal mathematics. He turned away from the mysterious to the mathematical, although he made the existence of matter depend upon God. His physics was a positive explanation of matter from which were banished the entities formerly in use, and in which nothing was accepted which would not be received in mathematics. He arrived at the conception of a complete revolution in physics by the substitution of mathematical explanations for scholastic formulas. According to Jascalevich, "Within the domain of scientific inquiry he looms up like a spectacle of intellectual fervour, supplying the cruel anæsthetics under which a petrified Scholasticism is to be carved out like a tumor from contemporary minds."[27]

Descartes was probably the first to attempt to give a complete mathematical and mechanical interpretation of the universe. The mathematical interpretation of nature, expressed simply in terms of mechanical law, was alien to the whole temper of the Middle Ages and to the Aristotelianism of his own age. Starting with extension and motion, Descartes attempted to develop a mechanical interpretation of the universe, free from the "substantial" forms and "occult" qualities which had been used in many previous explanations. He developed his theory of *vortices* to explain by means of extension, divisibility, and motion the entire universe, thinking of it as if it were a machine. Descartes' ideas concerning the universe continued to prevail on the continent for at least fifty years after Newton published his *Principia* in 1687. In 1730 Jean Bernoulli won the prize of the French Academy of Sciences for his defense of Descartes' vortices against Newton's "erroneous reasoning." Bernoulli declared:

> It would be a kind of ingratitude, if we did not recognize that it is principally to M. Descartes that we are thankful for the first ideas which he has given us in order to reason in physics, by principles which can be understood clearly, in place

[26] René Descartes, *Renatus Des-cartes excellent Compendium of Musick: with Necessary and Judicious Animadversions Thereupon* (London: H. Mosely, 1653), preface.

[27] A. A. Jascalevich, *Three Conceptions of Mind*, Dissertation Columbia University (New York: Columbia University Press, 1926), p. 73.

of all that trash ("fatras") of occult qualities, of substantial forms, of faculties, of plastic virtues, and of a hundred such chimeras that antiquity left to us.[28]

Before the time of Descartes the origin and maintenance of the physical universe had been explained by means of "souls." Descartes advanced the concept that the universe was derived and maintained by general laws. His theory of vortices had no validity, but it anticipated the more accurate theory of Newton. Newton made a clear distinction between experimental philosophy and philosophy in general, a distinction which Descartes never made.[29] The theory of universal gravitation was far more accurate and mathematical than the theory of vortices. Newton discredited Descartes and has in turn succumbed to Einstein, and who can say when Einstein's theory will be superseded by another?

7. *Practical Philosophy.* Descartes was not interested in developing practical philosophy, which may be taken to include the problems of ethics, esthetics, and politics. He did very little with ethics and nothing at all with esthetics and politics. Thus he wrote for the benefit of the Queen of Sweden:

> It is true that I have usually refused to write my thoughts concerning Morals ("la Morale"), and there are two reasons for this. First, there is no subject in which the malicious ones can more easily find pretects for slander; second, I believe that it belongs only to Sovereigns or to those who have authority, to interfere with the habits of others.[30]

With the detachment of a modern scientist, Descartes did not attempt to deal with the problems of practical philosophy. It seems probable that he thought that these matters could not be handled with mathematical exactness and certitude, and that therefore they should not be dealt with at all. Charpentier summed up very well his attitude on practical philosophy:

> The questions of ethics ("morale") are not, like the propositions of mathematics, simple questions of which the difficulty consists in discovering a determinate relation between a few data which are perfectly defined. In ethics the number of data is entirely unlimited, and the relations which connect these data cannot be fixed in an exact manner.
>
> Thus.... the moral and political sciences,.... did not appear to have, in the eyes of Descartes, the character of true sciences.[31]

[28] Jean Bernoulli, *Nouvelles Pensées sur le Système de M. Descartes* (Paris: Claude Jombert, 1730), p. 10.

[29] Cf. A. J. Snow, "Newton's Objections to Descartes' Astronomy," *The Monist*, XXXIV (1924), 544.

[30] Charles Adam and Paul Tannery, *Oeuvres de Descartes* (Paris: Leopold Cerf, 1897-1913), V, 86, 87.

[31] Thomas V. Charpentier, *Essai sur la Méthode de Descartes*, Thesis, University of Paris (Paris: Charles Delagrave, 1869), p. 170.

8. *Summary*. The influence of mathematics on the philosophy of Descartes may be summarized as follows:

1. Mathematics and philosophy were significantly related in the thinking of Descartes, though much of the "philosophy" with which he dealt would be considered "science" today.

2. Mathematics was related to all the major problems of philosophy as treated by Descartes. It was related *positively* to the problems of method, epistemology, metaphysics, and natural philosophy; it was related *negatively* to the problems of practical philosophy: ethics, esthetics, and politics.

3. The most significant phase of the interrelationship of mathematics and philosophy in the thinking of Descartes was the use of the technique of analysis.

4. Judging by Pythagoras and Descartes, mathematicians who philosophize tend to give objective reality to some phase of mathematics. Thus Pythagoras gave a reality to *number* and Descartes gave reality to *geometrical extension*.

The Teachers' Department

Edited by
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On Writing the General Term Coefficient of the Binomial Expansion to Negative and Fractional Powers, in Tri-Factorial Form

By WILLIAM FUNKENBUSCH
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In dealing with the expansion of $(a+b)^p$ most of our algebra texts give the coefficient of the $(k+1)$st term as:

$$\text{(I)} \qquad \frac{p(p-1)(p-2)\cdots\cdots\cdots \text{to } k \text{ factors}}{1.2.3\cdots\cdots\cdots k}.$$

Surprisingly few give the form:

$$\text{(II)} \qquad \frac{p!}{(p-k)!\,k!} \qquad (p>0,\ p \text{ integral})$$

which has two qualities which mathematicians strive for in their formulas; compactness and symmetry. The latter, the tri-factorial form, because of this, is better retained by our algebra students. It was a desire on my part to extend the latter form to include the cases where p became negative, fractional, or both, which led to a development of four formulas where the $(\pm n/d)!$'s of the formulas are by definition the p and $(p-k)$ factorials of (II).

The positive integral factorial field is of course already firmly established by

$$\text{(1)} \qquad m! = m(m-1)!$$

The coefficient of the $(k+1)$st term of the expansion of $(a+b)^s$ being

(2) $$\frac{s!}{(s-k)!\,k!}$$

and that of $(a+b)^{-t}$ being

(3) $$\frac{(k+1)(k+2)\cdots\cdots\cdots(k+t-1)}{(t-1)!}(-1)^k,$$

using (2) with $s=-t$ we get $(a+b)^{-t}$ giving as the coefficient of the $(k+1)$st term

(4) $$\frac{(-t)!}{(-t-k)!\,k!}.$$

From (3) and (4) then

(5) $$\frac{(-t)!}{(-t-k)!\,k!}=(-1)\frac{k(k+1)(k+2)\cdots\cdots\cdots(k+t-1)}{(t-1)!}$$ and

(6) $$\frac{(-t)!}{[-(k+t)]!}=\left|\frac{[(k+t)-1]!}{(t-1)!}\right|.$$

Set $t=p_1$ and $k+t=p_2$, then (6) becomes:

$$\frac{(-p_1)!}{(-p_2)!}=\frac{(p_2-1)!}{(p_1-1)!}.$$

Now a $p=f(p_1,p_2\cdots\cdots p_n)$ which will satisfy (7) may be defined by the relation

(8) $$|(-p)!|=[(p-1)!]^{-1}.$$

Thus is the negative integral field set up.

We see that the negative fractional field comes from (8) with $p=n/d$ or then

(9) $$|(-n/d)!|=[(n/d-1)!]^{-1}.$$

*Notice that the negative fractional factorial is defined in terms of a positive fractional factorial by (9), and the positive fractional factorial may be taken care of by (1) with $m=n/d$ or we have

(10) $$(n/d)!=n/d(n/d-1)!$$

For example $(4/3)!=4/3(1/3)!$, but what is the value of $(1/3)!$? Since the fractional factorials will always occur in pairs in writing the binomial

*Case $n<d$ is purposely avoided here to be taken up later.

coefficients in fractional factorial form and since they will always be an integral difference apart, the choice of the value of

$$(1/3)![\text{i. e. } (n/d)!, \quad n<d]$$

is purely arbitrary, but we will choose to say

$$(11) \qquad (n/d)! = n/d, \quad n<d$$

for the reason that such a choice will make

$$(12) \qquad |(-n/d)!| = 1, \quad n<d.$$

For example the third term coefficient of the expansion of $(a+b)^{-2/3}$ is

$$\frac{(-2/3)!}{(-8/3)!\,2!} = \frac{5}{9} \qquad \text{set } (-2/3)! = x \text{ and solve}$$

$$\frac{(5/3)!\,x}{2!} = \frac{5}{9} \qquad x = \frac{5.2}{9(5/3)(2/3)} = \frac{5.2.3.3.}{9.5.2} = 1.$$

Now singling out formulas (9), (10), (11) and (12) and noting that for case $n=d$ all four reduce to the identity $1 \equiv 1$, we have:*

if $n \geqq d$	if $n \leqq d$
1. $(n/d)! = n/d(n/d-1)!$	3. $(n/d)! = n/d$
2. $\lvert(-n/d)!\rvert = [(n/d-1)!]^{-1}$	4. $\lvert(-n/d)!\rvert = 1.$

These present a supposedly unique method of writing the coefficient of the general term of the expansion of $(a+b)^p$ with negative and fractional p's analogous to the case of the positive integral, which is decidedly pleasing in its treatment of the negative integral case in particular. For example:

Find the ninth term coefficient of the expansion of $(a+b)^{-4}$.
Standard form:

$$\text{(I)} \qquad \frac{(-4)(-5)(-6)(-7)(-8)(-9)(-10)(-11)}{1.2.3.4.5.6.7.8} = 165.$$

Tri-Factorial form:

$$\text{(II)} \qquad \frac{(-4)!}{(-12)!\,8!} = \frac{11!}{3!\;8!} = 11.5.3.$$

Of course cancelling can be done in (I) as in (II) but notice the compactness and symmetry of (II) as compared with (I).†

*$n>0$, $d>0$.

†Article by author "The Absolute Value of the Negative and Fractional Factorial". Proceedings of the Missouri Academy of Science. Vol. 6. No. 4 whole No. 20, March 25, 1941. (Note in this article mistake in printing, right side of equation (1) should be $(n/d)(n/d-1)!$.

Notes and Comments

Edited by
N. A. COURT

(NOTE 2)*

1. Let (A), (B) be two given circles, P and P' two points on (B), p, p' their powers for (A), and N, N' their projections upon the radical axis of (A), (B).

We have, both in magnitude and in sign,

$$p : p' = 2NP.AB : 2N'P'.AB = NP : N'P'.$$

Now if O is the trace of the line PP' on the radical axis of (A), (B), we have, both in magnitude and in sign,

$$OP : OP' = NP : N'P' = p : p',$$

hence: *The ratio of the powers, with respect to a given circle, of two points on a second circle is equal, in magnitude and in sign, to the ratio of the distances of those points from the point of intersection of the line joining them with the radical axis of the two circles.*

2. The point O is the center of a circle (0) orthogonal to both (A) and (B), and the points P, P' are inverse with respect to (0), hence the preceding proposition implies that *If two points are inverse for a circle, the ratio of their powers with respect to a second circle orthogonal to the first is equal, in magnitude and in sign, to the ratio of their distances from the center of the first circle.*

This proposition is due to Gh. Tzitzeica, Gazeta matematica, Vol. 1, p. 266, Art. 3, or Boletin Matematico (Buenos Aires), Vol. 14, 1941, p. 108 and pp. 230, 231.

3. The point O divides the segment PP' in the ratio $p : p'$, in magnitude and in sign, hence if (B) varies while P, P', (A) remain fixed, the point O will also remain fixed. We have thus a new proof of the known theorem that the radical axis of a fixed circle with a variable circle of a coaxal system passes through a fixed point, on the radical axis of the system. The proof is limited to the case of an intersecting system, but in the case of such a system it locates that fixed point with reference to the basic point of the system.

University of Oklahoma. N. A. COURT.

*The March Note is Note 1.

On Root Approximation

(Note 3)

In attempting to explain Horner's method to a student who knew nothing about the process of diminishing the roots of an equation, a process which is regarded as an essential part of Horner's method, the writer was led to a process which differs in form from Horner's method and also requires no elaborate theoretical basis. The basic idea was suggested by the method given by Dickson (Elementary Theory of Equations, p. 6) for computing the value of a polynomial $f(x)$ for a numerical value of x.

The process will now be applied to an example; detailed explanations will be given of only those steps which differ from the corresponding steps in Horner's method.

The equation

$$3x^3+x^2-5x-21=0. \tag{1}$$

has a root between 2 and 3 and probably nearer 2. Hence put

$$x=2+h, \tag{2}$$

so that h will have a value between 0 and 1 and probably nearer 0.

Now

$$\begin{aligned} 3x^3+x^2 &= 3(\ h+2)x^2+x^2 \\ &= (3h+1)x^2 \\ &+ (\quad +6)x^2 \qquad (3) \\ &= (3h+7)x^2. \qquad (4) \end{aligned}$$

Hence

$$\begin{aligned} 3x^3+x^2-5x &= (3h+7)(h+2)x-5x \\ &= (3h^2+7h-5)x \\ &+ (\qquad 6h+14)x \qquad (5) \\ &= (3h^2+13h+9)x. \qquad (6) \end{aligned}$$

Finally

$$\begin{aligned} 3x^3+x^2-5x-21 &= (3h^2+13h+9)(h+2)-21. \\ &= (3h^3+13h^2+\ 9h-21) \end{aligned}$$

$$
\begin{array}{lr}
(7) & +(\qquad 6h^2+26h+18) \\
 & (3h^3+13h^2+\ 9h-21) \\
(7) & +(\qquad 6h^2+26h+18) \\
(8) & = 3h^3+19h^2+35h-\ 3.
\end{array}
$$

Now the above work of obtaining an expression for $3x^3+x^2-5x-21$ in terms of h can be arranged as follows:

$$
\begin{array}{ll}
 & 3+\ 1-5-21\ \underline{|2} \\
(3) & \ \ +\ 6 \\ \hline
(4) & 3+\ 7 \\
(5) & \qquad 6+14 \\ \hline
(6) & 3+13+\ 9 \\
(6) & \qquad 6+26+18 \\ \hline
(8) & 3+19+35-\ 3
\end{array}
$$

Corresponding lines in the original work and in the contracted work are indicated by the same number on the left.

The equation

(9) $$3h^3+19h^2 35h-3=0$$

has a root near .1; hence put

(10) $$h=.1+k,$$

where k should be numerically less than .05. As above it is found that

(11) $$3k^3+19.9k^2+38.89k+.693=0,$$

the work being as follows:

$$
\begin{array}{l}
3+19.0+35.00-3.000\ \underline{|.1} \\
\ \ +\ \ .3 \\ \hline
3+19.3 \\
\qquad .3+\ 1.93 \\ \hline
3+19.6+36.93 \\
\qquad .3+\ 1.96+3.693 \\ \hline
3+19.9+38.89+\ .693
\end{array}
$$

The equation (11) has a root near $-.02$; hence put

$$k=-.02+m,$$

where m should be numerically less than $.005$. As above it is found that

(12) $3m^3+19.72m^2+38.0976m-.076864=0,$

the work being as follows:

$$\begin{array}{l}
3+19.90+38.8900+.693000 \;\underline{|-.02}\\
\quad-\quad .06\\
\hline
3+19.84\\
\quad-\quad .06-\quad .3968\\
\hline
3+19.78+38.4932\\
\quad-\quad .06-\quad .3956-.769864\\
\hline
3+19.72+38.0976-.076864
\end{array}$$

The equation (12) has $m=.002$ as approximate root. Hence an approximate root of (1) is

$$\begin{aligned}x&=2+h\\&=2.1+k\\&=2.08+m\\&=2.082.\end{aligned}$$

Louisiana State University. H. L. SMITH.

ERRATA

Volume XVII, No. 6

p. 232, line 13: for $z=1$ read $x=1$.
line 18: for a_4 read a_0.

p. 274, line 18: for series e, read series 3.
line 32: for derivatives, read derivative.

p. 275, last line: for the term $3b^4$, read $4b^4$.

Problem Department

Edited by
E. P. STARKE and N. A. COURT

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to EMORY P. STARKE, Rutgers University, New Brunswick, N. J.

SOLUTIONS

No. 435. Proposed by *V. Thébault*, Tennie, Sarthe, France.

Given the tetrahedron $ABCD$ and a point P. The lines PA, PB, PC, PD cut the faces BCD, CDA, DAB, ABC in A', B', C', D'. Find the locus of points P such that the volume of $A'B'C'D'$ is constant. Discuss the case $(A'B'C'D')=3(ABCD)$.

Solution by *C. E. Springer*, University of Oklahoma.

The solution can be simplified by using a coordinate system analogous to areal coordinates in the plane. (This might be called a "volumal" coordinate system.) Take for A, B, C, D the coordinates $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, and let P have coordinates (ξ^α), $(\alpha=1,2,3,4)$. Then X^α, given by

$$X^\alpha = X^\alpha_A + \lambda(\xi^\alpha - X^\alpha_A),$$

in which X^α_A are the coordinates of A, are the coordinates of an arbitrary point on the line PA. For point A', $X^1_A=1$ and $X^1=0$. Thus $\lambda = 1/(1-\xi^1)$, and the coordinates of A' are given by

$$(0,\ \xi^2,\ \xi^3,\ \xi^4)/(1-\xi^1).$$

Similarly, the coordinates of B', C', D' are given by

$$\frac{(\xi^1,\ 0,\ \xi^3,\ \xi^4)}{1-\xi^2},\quad \frac{(\xi^1,\ \xi^2,\ 0,\ \xi^4)}{1-\xi^3},\quad \frac{(\xi^1,\ \xi^2,\ \xi^3,\ 0)}{1-\xi^4}.$$

The volume of a tetrahedron with vertices $X^4_4, X^\alpha_\beta(\alpha,\ \beta=1,2,3,4)$ in "volumal" coordinates is the product of the determinant

$$|X^1_1\ X^2_2\ X^3_3\ X^4_4|$$

and the volume Δ of the tetrahedron of reference $ABCD$. Therefore

$$\text{Volume } A'B'C'D' = \Delta \cdot \begin{vmatrix} 0 & \xi^2 & \xi^3 & \xi^4 \\ \xi^1 & 0 & \xi^3 & \xi^4 \\ \xi^1 & \xi^2 & 0 & \xi^4 \\ \xi^1 & \xi^2 & \xi^3 & 0 \end{vmatrix} /(1-\xi^1)(1-\xi^2)(1-\xi^3)(1-\xi^4).$$

Thus the desired locus is the quartic surface

$$k(1-\xi^1)(1-\xi^2)(1-\xi^3)(1-\xi^4)+3\Delta\xi^1\xi^2\xi^3\xi^4=0.$$

In the special case where $k=-3\Delta$, the locus is the cubic surface

$$(\xi^1-1)(\xi^2-1)(\xi^3-1)(\xi^4-1)=\xi^1\xi^2\xi^3\xi^4.$$

The points A, B, C, D lie on both the quartic and cubic surfaces.

Solved also in Cartesian coordinates by *C. E. Springer* and by the *Proposer.*

No. 476. Proposed by *Nelson Robinson*, Louisiana State University.

The length of one arch of a hypocycloid is equal to the subtended arc of the fixed circle upon which the curve is generated. What is the relation between the radii of the fixed and rolling circles?

Solution by *Richard K. Thomas*, Long Island City, N. Y.

We find two consecutive points of intersection of the two curves and equate the two arcs taken between the two points. Thus

$$(1) \qquad 2\pi a = \int_0^{2\pi a/r} \sqrt{(dx/d\varphi)^2+(dy/d\varphi)^2}\cdot d\varphi,$$

where a is the radius of the rolling circle, r is the radius of the fixed circle, and the limits of integration correspond to the points of intersection obtained by solving simultaneously $x^2+y^2=r^2$ and

$$(2) \qquad x=(r-a)\cos\varphi+a\cos\{(r-a)\varphi/a\},$$

$$y=(r-a)\sin\varphi-a\sin\{(r-a)\varphi/a\},$$

the equations of the hypocycloid. Further we obtain from (2)

$$(dx/d\varphi)^2+(dy/d\varphi)^2=2(r-a)^2\{1-\cos(r\varphi/a)\}=4(r-a)^2\sin^2(r\varphi/2a),$$

whence (1) becomes

$$2\pi a = 2(r-a)\cdot\int_0^{2\pi a/r}\sin(r\varphi/2a)\cdot d\varphi = 8a(r-a)/r$$

and thus $$a : r = (4-\pi)/4.$$

Also solved by *Paul D. Thomas.*

No. 478. Proposed by *Howard D. Grossman*, New York City.

If on each side of triangle I as base, an equilateral triangle be constructed outwardly, the centroids of these three equilateral triangles themselves form an equilateral triangle II. If the original equilateral triangles be constructed inwardly, their centroids form another equilateral triangle III. Show that the difference of the areas of II and III is equal to the area of I.

Solution by *V. C. Bailey, W. B. Clarke, D. L. MacKay* and *Paul D. Thomas.*

The configuration considered in this question has been dealt with in MATHEMATICS NEWS LETTER, proposals 51 and 143 (May, 1934 and October, 1937) and in NATIONAL MATHEMATICS MAGAZINE, proposal 332 (Vol. 14, May, 1940, p. 482). In connection with the latter question it was shown that the squares of the sides of triangles II and III are given by

$$(a^2+b^2+c^2\pm 4\sqrt{3}\,S)/6,$$

where a b, c, are the sides and S the area of I. It follows that the areas of II and III are

$$(a^2+b^2+c^2)\sqrt{3}/24\pm\tfrac{1}{2}S,$$

hence the proposition.

Also solved analytically by *C. E. Springer.*

EDITORIAL NOTE. The figure formed by three equilateral triangles surmounting the sides of a basic triangle seems to have been first considered by D. Ch. L. Lehmus in *Crelle's Journal*, v. 50, 1855, p. 266, in connection with Fermat's problem (to which no reference is made): To find a point the sum of whose distances from the vertices of a triangle is a minimum. The figure has since been recurring in the mathematical periodicals. Numerous properties of this figure are listed in *Mathesis*, 1889, p. 188. See also *American Mathematical Monthly*, v. 25, 1918, p. 179.

No. 480. Proposed by *E. P. Starke*, Rutgers University.

Andy and Bill have five pennies each. These they match until one of them loses all. At each move the chances are even that Andy loses a penny to Bill or wins one from Bill. Find (a) the average number of moves required, (b) the most likely number of moves, and (3) the number, M, of moves such that the play is more likely to end on or before M moves than after, and is more likely to end on or after M moves than before.

Solution by the *Proposer.*

Let the probability that Andy has h pennies at the end of n moves be represented by $P(n,h)$. Of course Bill has then $10-h$, and always $P(n,h)=P(n,10-h)$. Thus for example, $P(5,10)=P(5,0)=1/32$, $P(5,8)=P(5,2)=5/32$, etc. Suppose $P(n,0)=x$, $P(n,2)=y$, $P(n,4)=z$ $=P(n,6)$, where n is odd. (The play cannot end with an even number of moves.) Since the number of coins is increased or decreased one at each move, we have, remembering that the game is terminated as soon as h is 0 or 10,

$$P(n+2,0)=\tfrac{1}{2}P(n+1,1)=\tfrac{1}{2}[\tfrac{1}{2}P(n,2)]=\tfrac{1}{4}y,$$

$$\begin{aligned}P(n+2,2)&=\tfrac{1}{2}P(n+1,1)+\tfrac{1}{2}P(n+1,3)\\&=\tfrac{1}{2}[\tfrac{1}{2}P(n,2)]+\tfrac{1}{2}[\tfrac{1}{2}P(n,2)+\tfrac{1}{2}P(n,4)]=\tfrac{1}{2}y+4z.\end{aligned}$$

$$\begin{aligned}P(n+2,4)&=\tfrac{1}{2}P(n+1,3)+\tfrac{1}{2}P(n+1,5)\\&=\tfrac{1}{2}[\tfrac{1}{2}P(n,2)+\tfrac{1}{2}P(n,4)]+\tfrac{1}{2}[\tfrac{1}{2}P(n,4)+\tfrac{1}{2}P(n,6)]=\tfrac{1}{4}y+\tfrac{3}{4}z.\end{aligned}$$

Continuing similarly, we have easily

$$P(n+4,0)=(2y+z)/16, \qquad P(n+4,2)=(5y+5z)/16,$$

$$P(n+6,0)=(5y+5z)/64,$$

which satisfy the recurrence formula

$$16P(n+6,0)=20P(n+4,0)-5P(n+2,0), \tag{1}$$

with $P(3,0)=0$, $P(5,0)=1/32$. The desired probability that Andy or Bill loses all at the end of n moves is $P_n=P(n,0)+P(n,10)=2P(n,0)$ as given in the table

n	5	7	9	11	13	15	17
P_n	2^{-4}	$5\cdot 2^{-6}$	$20\cdot 2^{-8}$	$75\cdot 2^{-10}$	$275\cdot 2^{-12}$	$1000\cdot 2^{-14}$	$3625\cdot 2^{-16}$

and so on.

Evidently (b) is answered by either 7 or 9 moves, for which the probability is 5/64, greater than that for any other specified number of moves. (From (1), whenever $P(n+4,0)$ is less than $P(n+2,0)$, then $P(n+6,0)$ is less than $P(n+4,0)$.)

The solution for (c) is found by adding enough of the first values of P_n to produce a probability of $\frac{1}{2}$. We find

$$\sum_{n=5}^{17} P_n = \frac{62322}{131072} < \tfrac{1}{2}, \qquad \sum_{n=5}^{19} P_n = \frac{275538}{524288} > \tfrac{1}{2}.$$

Thus the desired value is $M = 19$.

The average number (a) of moves is given by T where

$$T = 5\cdot 2^{-4} + 7\cdot 5\cdot 2^{-6} + 9\cdot 20\cdot 2^{-8} + 11\cdot 75\cdot 2^{-10} + 13\cdot 275\cdot 2^{-12} + \cdots .$$

To evaluate this recurring series* we note that

$$16(16 - 20x^2 + 5x^4)^{-1} = 1 + 5x^2/4 + 20x^4/16 + 75x^6/64 + \cdots$$

with coefficients satisfying (1). If we multiply both sides of this identity by x^5 and differentiate, we obtain

$$\begin{aligned} &80x^4(16 - 12x^2 + x^4)(16 - 20x^2 + 5x^4)^{-2} \\ &\qquad = 5x^4 + 7\cdot 5x^6/4 + 9\cdot 20x^8/16 + 11\cdot 75x^{10}/64 + \cdots . \end{aligned}$$

Divided by 16 and with $x = 1$ this gives $T = 25$, whence the required average number of moves is 25.

It is worth noting that in many problems on probability, as in this one, a number analogous to (c) is of more practical significance than the simple average (a). Note also that the problem is easily generalized to arbitrary numbers of coins and also to more players. See, for example, *American Mathematical Monthly*, problem 4003, August, 1941, p. 483.

No. 487. Proposed by *W. N. Huff*, The Hill School, Pottstown, Pa.

Find the volume generated by revolving a cube about a diagonal.

Solution by *Frank Hawthorne*, Alliance College, Cambridge Springs, Pa.

The generated figure may be considered in three parts, two of which are right circular cones and the third a hyperboloid of revolution of one sheet, whose rulings are the various positions of the two edges which do not meet the given diagonal. Since the hyperboloid is a ruled surface, the prismoidal formula may be applied to it.

*See, e. g., Hall and Knight, *Higher Algebra*, pp. 267-272.

Let e be the edge of the cube, so that its diagonal is $e\sqrt{3}$. Then, since the cosine of the angle between an edge and a diagonal to the same vertex is $1/\sqrt{3}$, the diagonal is trisected by the projections of the vertices upon it. Each cone has altitude $e\sqrt{3}/3$ and slant height e, whence its volume is $2\pi e\sqrt[3]{3}/27$. The perpendicular distance $(e\sqrt{6}/3)$ from a vertex upon the diagonal is the radius of either end section of the prismoid, while the distance $(e\sqrt{2}/2)$ between the midpoint of the diagonal and the midpoint of a remote side is the radius of the mid-section of the prismoid. Thus the volume of the hyperboloid is

$$\frac{1}{6}\,\frac{e\sqrt{3}}{3}\left\{2\pi\left(\frac{e\sqrt{6}}{3}\right)^2+4\pi\left(\frac{e\sqrt{2}}{2}\right)^2\right\}=\frac{5\pi e^3\sqrt{3}}{27}.$$

Therefore the total volume of two cones and hyperboloid is

$$\frac{e^3\pi\sqrt{3}}{3}.$$

Also solved by *D. L. MacKay*, *Paul D. Thomas*, and the *Proposer*.

No. 488. Proposed by *John Bristow*, student, Colgate University.

Two parabolas with axes perpendicular pass through the point P, (2,8) and have there the same tangent line and the same circle of curvature. If the equation of one of them is $y=6x-x^2$, find the equation of the other.

Solution by *J. Ernest Wilkins, Jr.*, Tuskegee Institute.

Since the axes of the parabolas are perpendicular, the desired equation is of the form $x=ay^2+by+c$. Since they have the same tangent and circle of curvature at P, the derivatives dy/dx and d^2y/dx^2 for the two parabolas must be equal at P. Hence also the derivatives dx/dy and d^2x/dy^2 are equal. They are respectively

$$1/(dy/dx)=1/(6-2x)=1/2, \quad dx/dy=2ay+b=16a+b\ ;$$

$$-(d^2y/dx^2)/(dy/dx)^3=-(-2)/8=1/4, \quad d^2x/dy^2=2a.$$

Therefore, $a=1/8$, $b=-3/2$. We find c be demanding that the parabola $x=ay^2+by+c$ pass through the point (2,8). Thus $c=6$, and the desired equation is $8x=y^2-12y+48$.

Also solved by *W. H. Bradford*, *Paul D. Thomas*, and the *Proposer*.

PROPOSALS

No. 516. Proposed by *V. Thébault*, San Sebastian, Spain.

The lines joining a given point L to the vertices A, B, C, D of a tetrahedron $(T)=ABCD$ meet the circumsphere of (T) again in the points A', B', C', D'. Let M, M' be the isogonal conjugates of L for the tetrahedrons (T), $(T')=A'B'C'D'$; let P, Q, R, S and P', Q', R', S' be the projections of M and M' upon the faces of (T) and (T'). Show that the tetrahedrons $PQRS$ and $P'Q'R'S'$ are similar.

No. 517. Proposed by *V. Thébault*, San Sebastian, Spain.

Consider triangles with integral sides such that the perimeter and the area are measured by the same number. Find all such triangles if further (1) two sides are consecutive integers; the same if (2) the three sides form an arithmetic progression.

No. 518. Proposed by *H. Jones*, United States Military Academy.

Find the envelope of a family of circles which touch one of two fixed perpendicular lines and intercept a segment of given length on the other.

No. 519. Proposed by *K. J. Nielsen*, Louisiana State University.

How many different arrangements can be made of two identical sets of n distinct cards if they are placed equally spaced around a ring?

No. 520. Proposed by *N. A. Court*, University of Oklahoma.

Two circles (A), (B) intersect in the points E, F. Two lines passing through E intersect (A), (B) again in points which determine the chords c, d. The parallels to c, d through E meet (B), (A) in the points Q, P, respectively. Show that the four points P, Q, F, cd, are collinear.

No. 521. Proposed by *Daniel Arany*, Budapest, Hungary.

Show that $$\int_0^1 \cos i\pi x \cdot \cos^n \pi x \cdot dx = 2^{-n} \binom{n}{k},$$

where $2k=n+i$.

No. 522. Proposed by *Daniel Arany*, Budapest, Hungary.

Show that

$$\int_0^1\int_0^1 \cos i\pi x\cdot\cos j\pi y[\cos\pi x+\cos\pi y]^n\cdot dx\cdot dy=2^{-n}\binom{n}{k}\binom{n}{m},$$

where $2k=n+i+j,\quad 2m=n+i-j.$

Bibliography and Reviews

Edited by
H. A. SIMMONS and P. K. SMITH

Plane Trigonometry (with) *Logarithmic and Trigonometric Tables.* By E. Richard Heineman. New York and London, McGraw-Hill Book Company, Inc., 1942. xii+167, 75 pages. $2.00; without tables, $1.50, separate tables, $.75.

This excellent and teachable text, made up of ten chapters or 87 articles, introduces the definitions of the trigonometric functions for a general or "trigonometric" angle and specializes these a little later to the case of an acute angle; the slight difficulty in this specialization for angle B on page 13 gives the instructor an early opportunity to act as a "trouble shooter" as the author suggests he should in his opening note to the student. Most of the usual topics are taken up, including inverse trigonometric functions and complex numbers, but excluding polar coordinates and the reduction formulas for functions of the sum of a general angle and a multiple of 90 degrees; the need of these formulas is felt in the proof of the general formula for sin $(A+B)$ which is given in the appendix. The author gives interesting expression to his thoughts in language ranging from informalities such as "multiplying top and bottom" in reducing fractions and "curved arrow" in describing angles, to technicalities such as "permissible values", "hypothetical decimal point", and "quadrille paper".

There are about 90 figures, more than 100 solved examples and illustrations, and 50 sets containing about 1200 exercises, nearly 100 of which are of the true false variety. These occupy roughly 9, 23, and 28 per cent respectively of the total space in the text. The index contains over 200 entries with more than 250 references to pages. Answers are given for about 500 of the odd numbered exercises. More than 50 footnotes are included; these emphasize important points or clarify usually bothersome situations rather than cite references. Four tables include a four-place table of natural functions for angles expressed both in radians and in degrees and minutes, two five-place tables of logarithms—one for numbers and the other for trigonometric functions, and a table of squares, cubes, square roots, cube roots and reciprocals of the integers from 1 to 100. The 4x6 $^7/_8$ in. columns of printed matter on 6x9 inch pages present a pleasing appearance. A line above the body of the text separates it from page numbers, chapter headings and numbers on left hand pages and from article numbers, topics and page numbers on right hand pages.

The comments which follow are about widely separated specific situations. None of these is of great importance in the appraisal of the text as a whole. "The distance r is always positive" (pp. 2-3) causes a little difficulty in the specification of points in polar coordinates in later courses. Figure 10 is slightly inaccurate; the cotangents of $+10$ and -10 degrees read from this figure appear to differ numerically, for instance. Some teachers prefer the author's emphatic denial (pp. 9-10) of the existence of csc 180° for example, but others like to give such matters a little more consideration. The true-false questions 33, 34 on page 11 are not indexed. Powers of the trigonometric functions are well explained (p. 16). One of the examples 3 or 4 (pp. 19-20) might well have required the use of an odd increment to make the result even. The second sentence in article 28 might mislead the unwary student, but the careful calculation of 1 radian a few lines later is desirable. The definition of the period of a function as the

"smallest" period (p. 54) is at variance with usual practice. Use of graphs for sorting out proper values (pp. 59-61) adds clarity to the solution of equations. Mention of the fact that 180 degrees is a solution of the equation in article 43 would have strengthened the important concluding sentence of that article. The words "and negative values of a" in the last sentence of article 46 are over optimistic in view of the two different results obtained in law (4) when $m=n=a=-2$. Locating decimal points of numbers and finding characteristics of logarithms (pp. 90-92) both may be accomplished by a single rule equivalent to that in the text but more simply stated. In the formal solution of right triangles, the cases in which angle B is given are under-emphasized in the exercises (pp. 24 and 105). The notations SAA, SSA, SAS, and SSS for specifying the nature of data for oblique triangles is good; in the formal exercies for the case SSA (p. 118), the given angle is always A. Commendable organization is shown in the proofs in articles 69 and 71. The suggestion for definition (p. 131) of the principal value of arcsecant might lead to difficulty when the student takes up the study of calculus, which is mentioned often throughout the text. Misprints which were noticed by or reported to, the reviewer are as follows: p. 3, line 13, the word "over" should be changed to "overly"; p. 3, line 15, "logarithms" is misspelled; p. 10, line 12, read "cos $\Theta = x/r$ $=$etc.;" p. 20, line 31, insert "l" in the word "probably"; p. 45, the last line ends with "$-$csc Θ"; p. 132, line 31, read "principal value etc." No incorrect answers were discovered.

The University of Omaha. JAMES M. EARL.

A Mathematics Refresher. By A. Hooper. Henry Holt and Company, Inc., New York, 1942. vii+342 pages. $1.90.

The American Edition of this book was preceded by an original (English) Edition which was written primarily for Royal Air Force candidates. Based on the notes and manuscripts compiled during twenty years of teaching, the book follows the author's own methods of explanation and teaching, "which, whatever their mathematical merit, did appear to achieve results," (p. v of the preface). In order to enable persons of average intelligence to clearly grasp the essentials of Arithmetic, Algebra, Geometry, and Trigonometry with a minimum of time and effort, it is necessary to break down the "largely artificial" barriers which traditionally separate the different branches of mathematics, that they may "dovetail into each other quickly and naturally," (p. v of the preface).

The nature of the decimal system and the use of decimal fractions are considered briefly (Chapters I and II); the metric system is adequately presented (Chapter VI); "Arithmetic Shorthand—Algebra" and "Shorthand Direction in Algebra" include a brief explanation of the index laws and a clear development of signed numbers (Chapters VII and VIII); "Geometry Without Tears" is devoted mainly to some elementary constructions and 13 geometric statements made without proof (Chapter IX); the treatment of statistical, speed, and proportion graphs deserves special mention (Chapter X); the topic of ratios introduces the representative fraction, percentage, and unitary analysis (Chapter XVII); a practical treatment of scale drawing includes the drawing of plans and maps, and gives specific directions for map making out-of-doors (Chapter XXIII); the parallelogram and triangle of velocities are considered and applied to air navigation (Chapter XXVIII); a very brief introduction to the idea of the Differential Calculus is presented (Chapter XXXII). The book is concluded with sets of varied and well constructed exercises organized according to chapters, a number of tests, answers, tables of weights and measures, and an index.

The book is distinctly a "Refresher," and is not intended to be viewed with mathematical rigor as the criterion. A second purpose of the author is to help the layman who asserts that he "cannot do mathematics;" therefore, "if the average man finds that an orange helps him to grasp abstract ideas, let him have an orange." Apparently the book has had a decided popular appeal, "since during recent months it has taken its place among the best sellers.

The reviewer has found a few errors. For example, the per cent sign is omitted in one equation on page 137. Again, on page 204, the last formula in the box should read "$v^2=u^2+2as$."

Explanations are fully given, but they stress mechanical procedure and memorization rather than rationalization. The worked examples are numerous. Excellent diagrams, 190 in number, are generously supplied throughout the text. Good use is made of italics and bold-faced type. Written with enthusiasm and a warm human interest, the book proceeds in a happy and interesting manner. It has evidently achieved in good measure the purposes for which it was written.

Northern Illinois State Teachers College. NORMA STELFORD.

The Development of Informal Geometry. By Robert Coleman, Jr. New York: Bureau of Publications, Teachers College, Columbia University, 1942. xii+178 pages.

This subject is a relatively recent innovation in the mathematical curriculum in the United States. Hence a study of its development should be of interest to teachers of mathematics and those responsible for the formulation of the mathematical curriculum in the junior and senior high schools. A recent volume reporting such a study should be of value to both groups. The investigation is divided into three parts.

In the first part it traces the movement in Germany from the earliest signs found in the educational reforms initiated by Pestalozzi, Herbart, and Froebel, and carried forward by their followers. Step by step the writer leads up to the reforms advocated by a group of educators and mathematicians under the leadership of Klein at the beginning of the present century.

Part II shows how the movement was transferred from Germany to England, where it gradually gained recognition among teachers of mathematics until it was *generally adopted* as a result of the Perry movement in the first decade of the present century.

The third part of the study is concerned with the movement as it developed in the United States. Pestalozzi's teachings were transferred to this country from Germany and from England and with it developed the work in informal geometry. Textbooks began to appear as early as 1830, but a most far-reaching influence was the report of the *Committee of Ten* in 1893. Then in 1902 Moore endorsed the Perry movement and proposed instruction in informal geometry. In 1912 the *Committee of Fifteen on Geometry* emphasized further the value of the teaching of informal geometry in the grades. Next came the demand for instruction in geometry in the junior high schools and finally the recommendations of the *National Committee on Mathematical Requirements* in 1923.

The most recent recommendation is that of the *Joint Commission of the Mathematical Association of America* and the *National Council of Teachers of Mathematics* in 1940.

Besides the tracing of the historical development, the discussion of the objectives of informal geometry will be of interest to the reader. Three groups of people, not necessarily exclusive of each other, have advocated instruction in informal geometry: (1) those whose primary interest has been in general elementary education, (2) those

whose interest was centered on the mathematics of the secondary school, and (3) those whose interest was in the practical applications of geometry in science and technology. The investigator concludes that there are five major objectives of the teaching of informal geometry: (1) cultivation of space perception, (2) acquisition of useful knowledge, (3) acquisitioning the pupil with his environment, (4) training in the methods of observation and discovery, (5) preparing the pupil to recognize the need for logical proof.

The study is a scholarly piece of work, worthy of the attention of the teachers of mathematics.

University of Chicago. E. R. BRESLICH.

Differential Equations. By Ralph Palmer Agnew. McGraw-Hill Book Company, New York, 1942. vi+341 pages.

This book of sixteen chapters (plus an appendix on beam problems) differs from the usual texts offered for a first course on differential equations in that: (1) it is designed for a full year course; (2) it incorporates a statement and discussion of several topics of elementary and advanced calculus necessary for rigorous discussion of differential equations; (3) problems which are exercises in algebraic manipulation rather than in differential equations are purposely avoided; (4) emphasis is placed on a careful study of the validity of the methods used in obtaining solutions, not on the purely formal aspects; (5) partial differential equations are mentioned briefly in connection with boundary or initial conditions.

Linear differential equations with constant coefficients are attacked from several directions including Fourier series, variation of parameters, operational calculus. Chapters 9, 10, 11 cover applications to mechanics and electric circuit problems. Chapter 12 gives an introduction to a variety of topics such as exact linear equations, adjoint equations, eigenvalue problems, orthonormal sets. Chapters 15, 16 are devoted to Picard's method of successive approximations.

This text should be required reading for all teachers of differential equations if for no other reason than to cause them to question some of the devices used in other books. The discussion of inverse D operators is unusually careful and complete. It should be considered for all students majoring in mathematics. The better engineering students could also profit by referring to it.

Virginia Military Institute. W. E. BYRNE.

Spherical Trigonometry with Naval and Military Applications. By Keels, Kern, and Bland. McGraw-Hill Company, New York, 1942. xiii+163 pages, +118 pages of tables.

The three authors of this text come from the United States Naval Academy. As would be expected, the exercises throughout are replete with the spirit and language of naval and military matters.

Chapter I covers *logarithms.* This is a well organized chapter. The authors include exercises often used—in the calculus and differential equations—in which expressions including sums and differences of logarithms are required to be written as products and quotients. Included at the end of this chapter is a set of forty exercises and problems. Many of the problems are illustrated by pictures of airplanes and ships.

Chapter II gives the formulas, and a review, of plane trigonometry. Twenty-six problems on plane trigonometry are included at the end of this chapter.

In Chapter III the spherical trigonometry of the right triangle is covered. At the end of this chapter the oblique triangle in which two sides and their included angle is given is solved by right triangles. It is good to introduce the student early to this most important case used in navigation.

Elementary Applications of the right spherical triangle is the subject of Chapter IV. In the first of the chapter the earth coordinates and the terrestrial triangles are explained. Emphasis is placed on finding a great circle distance between two points, and also the courses of arrival and departure when given the latitudes and longitudes of the points of departure and arrival. In this chapter plane sailing, parallel sailing, and middle latitude sailing are treated. The chapter ends with a discussion of the Mercator chart. In a footnote, the method of finding a meridional part—assuming a meridian to be a circle—is found by the calculus.

Chapter V is devoted to an ample development of the formulas of the spherical trigonometry of the oblique triangle.

The last chapter of the text, the sixth, is on applications. In the first of the chapter the haversine formula is applied to the terrestrial triangle case of two sides and the included angle. The celestial sphere and the astronomical triangle are discussed next. The important case of finding the altitude and azimuth of a heavenly body from its hour angle, declination, and the latitude of the observer is treated. The methods of determining the latitude by a transit of the meridian and the longitude by a "time sight" are given before presenting the modern method, in the last of the chapter, for the determination of latitude and longitude by the Sumner lines. The two useful methods, by Ageton and Driesonstok, for finding the altitude and azimuth when the hour angle, declination, and latitude of the observer are known, are given in this chapter.

The text contains an appendix of four parts: Part A contains a treatment of the mil; Part B, the theory and use of the range finder; Part C, a method of solving a spherical triangle graphically by stereographic projection; and Part D, the principle of the maneuvering board.

Answers to all problems are furnished. The text has a set of very complete tables, including five-place logarithms of numbers and the trigonometric functions, five-place natural tables of the usual trigonometric functions, and a table of natural haversines and their logarithms.

In this text—as with a great number of trigonometry texts—the question may be raised about the use of approximate numbers: why do the authors not exercise more care in rounding off answers to a degree of approximation not to exceed the accuracy of the least accurate of the numbers representing the measurements of the original data? The answers on page 19 illustrate the point. The answer to the Example on page 61 is 73° 24′, not 73° 20.5′ Perhaps a figure showing the departure measured along a parallel near the middle latitude would have helped the student appreciate the middle latitude formula as given on page 61.

The writer has not taught this text, but has used it as a reference in teaching navigation. It has been very helpful in this connection. After mastering this text, a student should sail through a text like Dutton's Navigation and Nautical Astronomy with little difficulty. This is a beautifully written textbook by three men who must surely know their field; the text is distinctive and superior.

The figures and pictures are splendid, and the format is excellent.

Louisiana Polytechnic Institute. P. K. SMITH.

Basic Mathematics for Pilots and Flight Crews. By C. V. Newsom and Harold D. Larsen. Prentice-Hall, Inc., New York, 1943. xi+153 pages. $1.50.

This book was written to cooperate with the Civil Aeronautics Administration and the United States Office of Education in "air-conditioning" the young people of this country. It should remove weaknesses of types that have been encountered by instructors in the United States Air forces. The nature of the work, which is of high school level, is, to some extent, suggested by its eleven chapter headings. These are: 1, *review of arithmetic;* 2, *elementary operations in algebra;* 3, *equations and formulas;* 4, *proportion and variation;* 5, *graphs;* 6, *units of measure and dimensional relations;* 7, *angular measure;* 8, *scales;* 9, *vectors and vector diagrams;* 10, *the circular slide rule in navigation;* 11, *trigonometry of the right triangle.*

The book also contains a 2-page table of natural values of sine, cosine, and tangent of certain angles on the interval from 0° to 90°; an appendix of "Miscellaneous Facts for Reference", which is highly informative; answers to selected problems; an index; and a circular slide rule, whose use is explained in Chapter X.

We believe that this book should be widely used, both during, and immediately following, the present war, in high schools and other preparatory schools. The practicability of the material here should arouse the interests not only of pilots and flight crews, but of many students who, apart from what they wish to do in life, would like to appreciate the real meaning of numerous mathematical concepts. For example, *vectors, parallelogram addition of vectors,* and *triangle addition of vectors,* as presented in this book, are sure to appeal to students.

Northwestern University. H. A. SIMMONS.

Mathematical Recreations. By Maurice Kraitchik. Norton, New York, 1942. 328 pages. $3.75.

This book is an improved edition of the author's *La Mathematique des Jeux,* published in 1930 in Brussels. It contains good discussions of certain more or less customary types of problems, for example, the Josephus Problem and problems about measuring wine, getting across rivers, shunting trains, dividing inheritances, and determining numbers that other people think of under given conditions. There is explanation of general principle underlying the puzzles. From theory of numbers there are discussions of special kinds of numbers, Fermat, Mersenne, figurate, perfect, cyclic, automorphic, and prime. Geometric problems include dissection of plane figures, mosaics, and simple topology. Forty pages are devoted to magic squares and twenty-five to probabilities. Arrangements and tours on the chess board and other problems relating to games are discussed. A chapter on the calendar sets up a perpetual calendar. This is a book written with authority, by a mathematician, with much symbolism, It is a book for the enthusiast, and such a person can spend many enjoyable hours in reading and working the puzzles of this book. There is much material that could be used by mathematics clubs or for mathematics bulletin boards. The only fault that this reviewer has to find with the book is that the answers to its problems are given immediately after the statements of them, so that there is little chance for one to solve them unaided.

Wright Junior College. RUTH M. BALLARD.